

TWISTED K -THEORY, REAL A -BUNDLES AND GROTHENDIECK-WITT GROUPS

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ABSTRACT. We introduce a general framework to unify several variants of twisted topological K -theory. We focus on the role of finite dimensional real simple algebras with a product-preserving involution, showing that Grothendieck-Witt groups provide interesting examples of twisted K -theory. These groups are linked with the classification of algebraic vector bundles on real algebraic varieties.

In our recent paper [13], we compare the Witt group of an algebraic variety over \mathbb{R} with a purely topological invariant we called $WR(X)$, associated to a space X with involution. In our comparison, X is the underlying space of complex points of the algebraic variety, and the involution is induced by complex conjugation. We are able to compute $WR(X)$ by comparing it to classical equivariant topological K -theory [19], Atiyah's Real K -theory $KR(X)$ [1], and other familiar invariants.

The Witt group of skew-symmetric forms is approximated in a similar way in [13], using another topological invariant. Surprisingly, the computation of this invariant is more subtle, as it is linked with equivariant twisted K -theory (in the sense of [5] and [2]).

This paper gives a more systematic study of this equivariant twisted K -theory, a variant we think is of independent interest. Our emphasis will be on examples linked with finite dimensional simple \mathbb{R} -algebras and Grothendieck-Witt groups.

The first section of this paper develops the basic notions of equivariant twisted K -theory in a geometric way, adapting many classical arguments of topological K -theory to vector bundles which are modules over an algebra bundle. We don't claim too much originality here. Much of the recent theory has already been developed in an ad hoc way (see for instance [3], [9], [4] or the 2014 thesis of El-kaïoum Moutuou [14]).

In the second section, we study specific examples linked with Real vector bundles. We follow Atiyah's viewpoint [1], replacing \mathbb{C} with \mathbb{R} -algebras with involutions. The most important examples for us are "balanced" algebras, such as Clifford algebras, for which we can provide a simpler description of the theory.

Date: September 29, 2015.

Weibel was supported by NSA and NSF grants,

In the last section we link twisted K -theory to the Grothendieck-Witt groups of symmetric and skew-symmetric bilinear forms on Real \mathcal{A} -bundles. In particular, we show that the Grothendieck-Witt group of skew-symmetric bilinear forms is isomorphic to a “twisted KR -group” associated to the quaternion algebra \mathbb{H} . This group is different from the group of symplectic bundles defined by J. Dupont [6] in another context.

We are particularly grateful to Jonathan Rosenberg, who made important comments on the first version of this paper.

1. G - \mathcal{A} BUNDLES

In this section we present a variant of (G -equivariant) twisted K -theory associated to an algebra bundle \mathcal{A} on a G -space X . We begin with a quick review of the non-equivariant theory.

Let A be a fixed Banach \mathbb{R} -algebra with unit and X a paracompact space. We suppose given a locally trivial bundle \mathcal{A} of Banach \mathbb{R} -algebras on X , with fibers isomorphic (non-canonically) to A ; the structure group of continuous algebra automorphisms has the compact-open topology (also called the *strong* topology).

By an \mathcal{A} -*bundle* we mean a locally trivial bundle E of (left) \mathcal{A} -modules on X such that $\mathcal{A} \times_X E \rightarrow E$ is continuous, and each fiber E_x is a finitely generated projective \mathcal{A}_x -module. Morphisms $E \rightarrow F$ are bundle maps whose fibers are module maps. The \mathcal{A} -bundles form an additive category and we write $K^{\mathcal{A}}(X)$ for the Grothendieck group of \mathcal{A} -bundles on X . When \mathcal{A} is a trivial algebra bundle, an \mathcal{A} -bundle is just a (classical) A -*bundle*, and we write $K^A(X)$ for $K^{\mathcal{A}}(X)$.

The simplest non-trivial examples arise when \mathcal{A} is an algebra bundle with fiber $A = M_n(\mathbb{C})$ and X is a finite CW complex. In this case, the Morita equivalence classes of possible \mathcal{A} are classified by their class in the topological Brauer group of X , which by a theorem of Serre ([7, 1.7]) is the torsion subgroup of $H^3(X; \mathbb{Z})$. Historically, these groups $K^{\mathcal{A}}(X)$ were the first examples of twisted K -theory (see [5], [12]). The theory has since developed to include the class in $H^3(X; \mathbb{Z})$ as part of the data (see [16]), with the awareness that the generalization to modules over a bundle of Banach algebras gives rise to interesting applications in physics, as mentioned for example in [20].

Now let G be a compact Lie group acting on X . As usual, we say that G acts on a bundle E if G acts on E compatibly with the structure map $E \rightarrow X$, in the sense that multiplication by $g \in G$ sends E_x to E_{gx} . If \mathcal{A} is a bundle of Banach algebras with fiber A , we say that G acts on \mathcal{A} if G acts on the underlying bundle of \mathcal{A} by algebra isomorphisms, i.e., if multiplication by any $g \in G$ is an algebra isomorphism $\mathcal{A}_x \rightarrow \mathcal{A}_{gx}$ for each $x \in X$. The notion of a G - \mathcal{A} bundle is somewhat related to Fell’s Banach $*$ -algebraic bundles over G [8].

Definition 1.1. Let E be an \mathcal{A} -bundle whose fiber is a finitely generated projective A -module. We say that E is a G - \mathcal{A} bundle on X if G acts on E (and \mathcal{A}) so that

$$g(a \cdot e) = g(a) \cdot g(e), \quad \forall g \in G, x \in X, a \in \mathcal{A}_x, e \in E_x.$$

We write $K_G^{\mathcal{A}}(X)$ for the Grothendieck group of the additive category of G - \mathcal{A} bundles. When G acts on A and \mathcal{A} is the trivial algebra bundle $X \times A$, we will write $K_G^A(X)$ for $K_G^{\mathcal{A}}(X)$.

The group $K_G^{\mathcal{A}}(X)$ is contravariantly functorial in the variables G and X . It is also covariant in \mathcal{A} (and contravariant for finite flat maps $\mathcal{A} \rightarrow \mathcal{A}'$).

The groups $K_G^{\mathcal{A}}(X)$ are an equivariant version of twisted K -theory [12] [4]. The prototype of this construction is when \mathcal{A} is an algebra bundle with fiber $M_n(\mathbb{C})$ and G acts trivially on \mathbb{C} . In this case, Atiyah and Segal [3] have shown that G -algebra bundles are classified up to Morita equivalence by their class in the *equivariant Brauer group* $\text{Br}_G(X)$, which is the torsion subgroup of $H^3(EG \times_G X, \mathbb{Z})$.

Examples 1.2. Suppose that \mathcal{A} is a trivial bundle with fiber A .

a) When G acts trivially on A , a G - \mathcal{A} bundle is just an A -linear G -bundle. If A is a finite simple algebra, $K_G^{\mathcal{A}}(X)$ is the usual equivariant K -group $KO_G(X)$, $KU_G(X)$ or $KSp_G(X)$, depending on A .

b) When \mathcal{A} is the trivial bundle with fiber \mathbb{C} , and G is the cyclic group $\text{Gal}(\mathbb{C}/\mathbb{R})$, a G - \mathcal{A} bundle is the same thing as a Real vector bundle in the sense of Atiyah [1], and our $K_G^{\mathbb{C}}(X)$ is Atiyah's $KR(X)$.

c) When \mathcal{A} is the trivial bundle with fiber \mathbb{H} , and G is a finite subgroup of \mathbb{H}^\times acting by inner automorphisms on \mathbb{H} , the notion of G - \mathcal{A} bundle seems new. We call these *Real quaternionic bundles*; see Examples 2.12 and 2.16. We will see in Theorem 3.5 and Example 3.7 that this case is related to the Grothendieck-Witt group of skew-symmetric bilinear forms on vector bundles over X .

d) (Morita invariance). If G acts on \mathcal{A} then G acts slotwise on $M_n(\mathcal{A})$, and the Morita equivalence of \mathcal{A} -bundles and $M_n(\mathcal{A})$ -bundles extends to an equivalence between the categories of G - \mathcal{A} bundles and G - $M_n(\mathcal{A})$ bundles. Thus $K_G^{\mathcal{A}}(X) \cong K_G^{M_n(\mathcal{A})}(X)$.

e) Suppose that G acts on A , and acts trivially on X , so that G acts on the trivial algebra bundle \mathcal{A} . Recall that the twisted group ring $A \rtimes G$ is the left A -module $A \times G$ with multiplication $g \cdot a = g(a) \cdot g$. In this case, we can form the trivial algebra bundle $\mathcal{A} \rtimes G$ with fiber $\mathcal{A} \rtimes G$, and a G - \mathcal{A} bundle on X is the same as an $A \rtimes G$ -linear bundle. Indeed, a left $A \rtimes G$ -module E is the same as a left A -module, with an action of G , satisfying the intertwining relation of Definition 1.1.

Many properties of ordinary vector bundles remain valid for G - \mathcal{A} bundles. For example, the kernel of a surjection $E \xrightarrow{s} E''$ of G - \mathcal{A} bundles is the subbundle whose fiber at x is the kernel of $E_x \rightarrow E''_x$.

Lemma 1.3. *Let $E \xrightarrow{s} E''$ be a surjection of G - \mathcal{A} bundles. Then the kernel E' of this map is a G - \mathcal{A} bundle, and*

$$0 \rightarrow E' \rightarrow E \xrightarrow{s} E'' \rightarrow 0$$

is a split exact sequence.

Proof. Clearly, E' is a G - \mathcal{A} subbundle of E . To split the short exact sequence, choose an arbitrary \mathcal{A} -bundle splitting $E'' \rightarrow E$; this may be done locally on X and the splittings may be combined using a partition of unity, as in the classical setting. Using a Haar measure on G , we can average this splitting to get an G -equivariant \mathcal{A} -module splitting t . Since $t \circ s$ is idempotent with kernel E' , the sequence is split exact. \square

When X is a point and G acts trivially on A , a G - \mathcal{A} module is just a finite $A[G]$ -module. If $A = \mathbb{C}$ then, as in Example 1.2(a), $K_G^{\mathbb{C}}(X)$ is the character ring $R(G)$ of G . This shows that a G - \mathcal{A} bundle need not be a summand of a fixed reference bundle $X \times A^n$.

We will show that every bundle is a summand of a different kind of “trivial” bundle. In our setting we define a “trivial” G - \mathcal{A} bundle to be a bundle of the type $\mathcal{A} \otimes M$, where M is a finite dimensional $\mathbb{R}[G]$ -module and G acts diagonally. The following theorem is borrowed from Segal’s paper on equivariant K -theory ([19, p. 134]).

Theorem 1.4. *Let E be a G - \mathcal{A} bundle on a compact space. Then there exists a G - \mathcal{A} bundle F such that $E \oplus F$ is isomorphic to $\mathcal{A} \otimes M$ for some finite dimensional G -module M .*

Proof. Let $\Gamma = \Gamma(E)$ be the topological space of continuous sections of E ; it is naturally a G -module. Let Γ' denote the union of its finite dimensional G -invariant subspaces. By a variant of the Peter-Weyl theorem, Γ' is a dense invariant subspace of Γ . Now, for each point x of X , we choose a finite set s_i^x of global sections such that the $(s_i^x)(x)$ generate E_x as an \mathcal{A}_x -module. Since Γ' is dense, we may choose the s_i^x in Γ' . By continuity, there is an open neighbourhood U_x of x such that the $(s_i)(y)$ generate E_y as an \mathcal{A}_y -module for any $y \in U_x$. Since X is compact, we only need the s_i^x for finitely many x ; they all lie in a fixed finite dimensional G -invariant subspace M of Γ . Thus there is a surjection $\mathcal{A} \otimes M \rightarrow E$. We conclude thanks to Lemma 1.3. \square

The usual argument shows that $K_G^{\mathcal{A}}$ is a homotopy functor. Let I be the unit interval; if X is a G -space, we regard $X \times I$ as the G space with $g(x, t) = (g(x), t)$. Given an algebra bundle \mathcal{A} on X , we abusively write \mathcal{A} for the pullback of \mathcal{A} along the projection $p : X \times I \rightarrow X$.

Theorem 1.5. *If X is a compact space, the projection $X \times I \xrightarrow{p} X$ induces an isomorphism*

$$p^* : K_G^{\mathcal{A}}(X) \cong K_G^{\mathcal{A}}(X \times I).$$

Proof. The proof is analogous to the one in the classical case [10, I.7.1]: if $p = p(t)$ ($t \in I$) is a continuous family of projection operators on a trivial bundle $\mathcal{A} \otimes M$, it is enough to show that $p(1)$ is isomorphic to $p(0)$. By compactness of X , each $t \in I$ has a neighborhood U such that for each $u \in U$ the operator $g = 1 - p(t) - p(u) + 2p(t)p(u)$ is invertible and $gp(u)g^{-1} = p(t)$. We conclude by using the compactness of I . \square

We can use Theorem 1.4 in order to prove an analogue of the Serre-Swan theorem in this framework. If B is a Banach algebra with a continuous G action, we define the equivariant Grothendieck group $K_G(B)$ to be the Grothendieck group of the category of finitely generated projective B -modules with a continuous G action. As in the case of bundles, we assume that the actions of B and G are intertwined: we have the relation

$$g(b \cdot e) = g(b) \cdot g(e).$$

Note that if G is finite then this definition is purely algebraic.

If E is a G - \mathcal{A} bundle, and $B = \Gamma(\mathcal{A})$ is the Banach algebra of sections of the algebra bundle \mathcal{A} , then Theorem 1.4 implies that the space of sections $\Gamma(E)$ is a finitely generated projective B -module with a continuous G -action.

Theorem 1.6. *If X is compact, the functor Γ defines an equivalence between the category of G - \mathcal{A} bundles on X and the category of finitely generated projective B -modules with a continuous G action.*

Proof. This is completely analogous to the usual proof of the Serre-Swan theorem [10, I.6.18]. The most difficult step is to show that Γ is essentially surjective; this is a direct consequence of Theorem 1.4. \square

Theorem 1.7. *Let G be a finite group and $B = \Gamma(\mathcal{A})$. If X is compact, Γ defines an equivalence between the category of G - \mathcal{A} bundles on X and the category of finitely generated projective modules over $B \rtimes G$. Thus*

$$K_G^A(X) \cong K_0(B \rtimes G).$$

Proof. It is easy to show via an averaging process that the category of finitely generated projective B -modules with a continuous G -action is equivalent to the category of finitely generated projective $B \rtimes G$ -modules. \square

When a compact Lie group G acts on a C^* -algebra B , Julg [9] showed that the equivariant K -theory of B , $K_G(B)$, is canonically isomorphic to $K_0(B \rtimes G)$, where $B \rtimes G$ is the twisted group ring. Taking $B = \Gamma(\mathcal{A})$ yields a more general version of Theorem 1.7.

Remark. When G is a finite group, A is a finite dimensional algebra with a G -action and $\mathcal{A} = X \times A$ is the trivial algebra bundle, then B is $C(X) \otimes A$, $C(X)$ being the ring of continuous functions on X .

In particular, if G acts trivially on A then the usual category of A -linear G -bundles is equivalent to the category of finitely generated

projective $(C(X) \rtimes G) \otimes A$ -modules, which is equivalent to the category of A -linear G -bundles; see Example 1.2(a).

Part (a) of the following purely algebraic theorem was originally proven by G.K. Pedersen [15, Thm. 35], using the notion of exterior equivalence of actions of a (locally compact) group on a C^* -algebra.

Theorem 1.8. *Suppose that a group G acts on a ring A by inner automorphisms $g(a) = x(g) a x(g)^{-1}$ via a representation $G \xrightarrow{x} A^\times$. Then*

- a) the twisted algebra $A \rtimes G$ is isomorphic to $A[G]$;*
- b) For every G -algebra C , $(C \otimes A) \rtimes G \cong (C \rtimes G) \otimes A$.*
- c) If G is a finite group and A is a finite dimensional Banach algebra, $K_G^A(X)$ is the equivariant K -theory of A -bundles of Example 1.2(a). In particular, $K_G^A(X)$ is independent of the representation x .*

Proof. Fix $g \in G$ and set $x = x(g^{-1}) = x(g)^{-1}$, $y(g) = g \cdot x$. The element $y(g)$ of $A \rtimes G$ commutes with every element of A :

$$y(g) \cdot a = gx \cdot a \cdot x^{-1}x = g(g^{-1}ag)x = a y(g).$$

The A -module map $A[G] \rightarrow A \rtimes G$ sending g to $y(g)$ is multiplicative: $y(g) \cdot y(h) = g \cdot x(g)^{-1}y(h) = gy(h)x(g)^{-1} = gh \cdot x(h)^{-1}x(g)^{-1} = y(gh)$.

Thus y defines a ring isomorphism $A[G] \xrightarrow{\cong} A \rtimes G$. This proves (a).

For (b), the map $(C \rtimes G) \otimes A \rightarrow (C \otimes A) \rtimes G$ sending $cg \otimes a$ to $(c \otimes a)y(g)$ is an isomorphism, because $y(g)c = g(c)y(g)$ in $(C \otimes A) \rtimes G$.

When G is finite and $\dim(A) < \infty$, Theorem 1.7 and part (b) imply that $K_G^A(X) = K_0((C(X) \rtimes G) \otimes A)$. That is, every G - \mathcal{A} bundle (with \mathcal{A} trivial) has the form $E \otimes_A A[G]$ for an A -bundle E , and $K_G^A(X)$ is isomorphic to the usual equivariant K -theory of A -bundles on X . \square

Example 1.9. Suppose that A is a finite dimensional simple \mathbb{R} -algebra and that G acts as the identity on the center of A ; This is the case for example if the center is \mathbb{R} . By the Noether-Skolem theorem, every automorphism on A is inner, i.e., conjugation by an element x . If the action lifts to a representation $G \rightarrow A^\times$, as in Theorem 1.8, then $K_G^A(X)$ is either $KO_G(X)$, $KU_G(X)$ or $KSp_G(X)$, depending on whether A is a matrix ring over \mathbb{R} , \mathbb{C} or \mathbb{H} .

Remark. The assumption that $x(g)x(h) = x(gh)$ is needed in Theorem 1.8. For instance, when $G = \{1, \tau\}$ is acting trivially on X , with $A = \mathbb{H}$ and $x(\tau) = i$, we show in Example 2.16 below that $K_G^{\mathbb{H}}(X)$ is isomorphic to $KU(X)$ and not $KSp(X)$.

2. REAL A -BUNDLES

We now restrict to the case when G is the cyclic group $\{1, \tau\}$ of order 2, that A is a Banach \mathbb{R} -algebra, and \mathcal{A} is an algebra bundle on X with fiber A . If there is an involution τ on \mathcal{A} which restricts to

algebra isomorphisms $\mathcal{A}_x \xrightarrow{\sim} \mathcal{A}_{\bar{x}}$, we call \mathcal{A} a *Real algebra bundle*. (This is a special case of G acting on \mathcal{A} in the sense of the previous section.)

We use the term “Real \mathcal{A} -bundle” for a G - \mathcal{A} bundle when $G = \{1, \tau\}$. Here is a paraphrase of Definition 1.1 in this setting.

Definition 2.1. Suppose that X is a space with involution τ , and \mathcal{A} is a Real algebra bundle on X . By a *Real \mathcal{A} -bundle* on X we mean an \mathcal{A} -bundle E together with an involution $\tau : E \rightarrow E$ which sends E_x to $E_{\bar{x}}$ and which is twisted \mathcal{A} -linear in the sense that $\tau(a \cdot e) = \bar{a}\tau(e)$ for $a \in \mathcal{A}_x$, $e \in E_x$. A morphism $\phi : E \rightarrow F$ of Real \mathcal{A} -bundles is a morphism of the underlying \mathcal{A} -bundles commuting with the involution ($\phi\tau = \tau\phi$).

Real \mathcal{A} -bundles form an additive category under Whitney direct sum of bundles, and we write $KR^{\mathcal{A}}(X)$ for its Grothendieck group. This group is contravariant in X and covariant in \mathcal{A} : given $\mathcal{A} \rightarrow \mathcal{A}'$, the functor $E \mapsto \mathcal{A}' \otimes_{\mathcal{A}} E$ defines a map $KR^{\mathcal{A}}(X) \rightarrow KR^{\mathcal{A}'}(X)$. Forgetting the involution yields a functor $KR^{\mathcal{A}}(X) \rightarrow K^{\mathcal{A}}(X)$.

Example 2.2. If A is equipped with an algebra involution $a \mapsto \bar{a}$, and \mathcal{A} is the trivial algebra bundle $X \times A$ with $\tau(x, a) = (\bar{x}, \bar{a})$, we call \mathcal{A} a *trivial Real algebra bundle*, and use the term *Real A -bundle* for a *Real \mathcal{A} -bundle*. Unless stated otherwise, every algebra bundle in the rest of this section will be a trivial Real algebra bundle.

For example, when the involution on A is trivial, a Real A -bundle on X is just an A -linear G -bundle, where $G = \{1, \tau\}$. As pointed out in Example 1.2(a), $KR^A(X)$ is $KO_G(X)$, $KU_G(X)$ or $KSp_G(X)$ when A is a matrix algebra over \mathbb{R} , \mathbb{C} or \mathbb{H} , respectively.

When $A = \mathbb{C}$ and the involution is complex conjugation, a Real A -bundle is a Real vector bundle in Atiyah’s sense [1]. As pointed out in Example 1.2(b), $KR^A(X)$ is Atiyah’s $KR(X)$. This example motivates our choice to adopt Atiyah’s notation KR from [1].

Variant 2.3. Suppose that A^0 is a Banach algebra and $A = A^0[G]$ (with $\overline{a + b\tau} = a - b\tau$). If G acts trivially on X , the category of Real A -bundles is equivalent to the category of A^0 -bundles, so $KR^A(X) \cong K^{A^0}(X)$.

Lemma 2.4. *Fix A and X as above. There is a faithful functor \mathcal{E} from the category of finitely generated projective left $A \rtimes G$ -modules to the category of Real A -bundles whose underlying A -bundle is trivial.*

Proof. Suppose that M is a finitely generated projective left $A \rtimes G$ -module. Then M is a finitely generated projective left A -module, endowed with an involution $\tau \cdot m = \bar{m}$ such that $\overline{\bar{a} \cdot m} = a \cdot \bar{m}$. Then $\mathcal{E}(M) = X \times M$ is trivial as an A -bundle, and the involution $\tau(x, m) = (\bar{x}, \bar{m})$ makes it a Real A -bundle. As $\mathcal{E}(M)$ is natural in M , \mathcal{E} is a functor. \square

Remark. Set $B \cong C(X) \otimes A$, where $C(X)$ is the ring of continuous functions on X . By Theorem 1.7, the category of Real A -bundles on X is equivalent to the category of finitely generated projective modules over $B \rtimes G$. In particular, $KR^A(X) \cong K_0(B \rtimes G)$.

The Real A -bundle $\mathcal{E}(A \rtimes G)$ is $X \times (A \rtimes G)$, endowed with the involution $\tau(x, a + b\tau) = (\bar{x}, \bar{a} + \bar{b}\tau)$. Given a morphism $\phi : \mathcal{E}(A \rtimes G) \rightarrow E$ of Real A -bundles, $e(x) = \phi(x, 1)$ is a section of E . We immediately obtain:

Corollary 2.5. *Given a Real A -bundle E , a section e of the underlying bundle uniquely determines a morphism $\phi : \mathcal{E}(A \rtimes G) \rightarrow E$ of Real A -bundles, by the formula $(x, a + b\tau) \mapsto (x, ae_x + b\bar{e}_{\bar{x}})$.*

The universal property in Corollary 2.5 justifies the terminology that Real A -bundles can be free.

Definition 2.6. We say that a Real A -bundle is *free* if it is a direct sum of copies of $\mathcal{E}(A \rtimes G)$, i.e., $\mathcal{E}(F)$ for a free $A \rtimes G$ -module F .

Lemma 2.7. *If X is compact, any Real A -bundle E is a direct summand of a free Real A -bundle.*

Proof. By Theorem 1.4, E is a summand of a Real bundle of the form $X \times (A \otimes \mathbb{R}[G]^n) = \mathcal{E}(A \rtimes G)^n$. \square

Example 2.8. The ring $A \rtimes G$ has two orthogonal idempotents, $e_+ = (1 + \tau)/2$ and $e_- = (1 - \tau)/2$. Both Ae_+ and Ae_- are left ideals of $A \rtimes G$, and $A \rtimes G = Ae_+ \oplus Ae_-$. Thus both Real A -bundles $\mathcal{E}(Ae_+)$ and $\mathcal{E}(Ae_-)$ have $X \times A$ as their underlying A -bundle, and

$$\mathcal{E}(A \rtimes G) = \mathcal{E}(Ae_+) \oplus \mathcal{E}(Ae_-).$$

The bundle $\mathcal{E}(Ae_+)$ has the usual involution $\tau(x, a) = (\bar{x}, \bar{a})$, while the bundle $\mathcal{E}(Ae_-)$ has the involution $\tau(x, a) = (\bar{x}, -\bar{a})$.

We will write A_σ for the left $A \rtimes G$ -module Ae_- , i.e., the left A -module A with the involution $\tau(a) = -\bar{a}$. Alternatively, A_σ is the Real A -module $A \otimes \mathbb{R}_\sigma$, where \mathbb{R}_σ is the sign representation of G .

In the rest of this section, we identify $KR^A(X)$ in some special cases. Recall that the cyclic group $G = \{1, \tau\}$ acts on a Banach algebra A .

Finite simple algebras

We will be primarily interested in Real A -bundles when A is a finite dimensional simple \mathbb{R} -algebra. Studying Real A -bundles for any finite dimensional *semisimple* \mathbb{R} -algebra A does not yield more generality. Since any finite dimensional semisimple \mathbb{R} -algebra with involution A is a product of simple algebras with involution A_i and algebras $A_j[G]$, every Real A -bundle is canonically a product of Real A_i -bundles and Real $A_j[G]$ -bundles. We leave the details to the reader.

Lemma 2.9. *If A is a central simple \mathbb{C} -algebra, and the involution is trivial on \mathbb{C} , then $KR^A(X) \cong KU_G(X)$.*

Proof. By Noether-Skolem, $\bar{a} = xax^{-1}$ for some $x \in A^\times$ with $x^2 \in \mathbb{C}$. Let $c \in \mathbb{C}$ be a square root of x^2 ; then $\bar{a} = (x/c)a(x/c)^{-1}$ and $(x/c)^2 = 1$. The result now follows from Theorem 1.8(b). \square

Example 2.10. Let A be the algebra $M_2(\mathbb{R})$ with the involution defined by conjugation by the diagonal matrix $j = (1, -1)$. That is, A is the Clifford algebra $C^{1,1}$ with $i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ satisfying $i^2 = -1$ and $\bar{i} = -i$, while $k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ satisfies $k^2 = +1$ and $\bar{k} = -k$. Since $j^2 = 1$, Theorem 1.8(b) yields $KR^A(X) \cong KO_G(X)$.

Lemma 2.11. *If A is a finite simple \mathbb{R} -algebra, $A \rtimes G$ is a semisimple \mathbb{R} -algebra.*

Proof. If V is a minimal left ideal of A , $\tau(V)$ is either V or disjoint from V by Schur's Lemma. In the latter case, consider the $A \rtimes G$ -modules $M = V \oplus \tau(V)$; either it is simple or has the form $W \oplus W'$ where $W = \tau(W)$ and $W' = \tau(W')$. Since A is a direct sum of minimal left ideals, this shows that A is a direct sum of simple left $A \rtimes G$ -modules, and hence that $A \rtimes G$ is semisimple. \square

Examples 2.12. If $A = \mathbb{C}$ and the involution is complex conjugation (resp., trivial) then $A \rtimes G$ is $M_2(\mathbb{R})$ (resp., $\mathbb{C} \times \mathbb{C}$).

If $A = \mathbb{H}$ and the involution is conjugation by i , then $A \rtimes G$ is $M_2(\mathbb{C})$. In this case, we call a Real \mathbb{H} -bundle a *Real quaternionic bundle* on X ; these came up in [13]. We will show in Example 2.16 below that if G acts trivially on X then the group $KR^A(X)$ is $KU(X)$.

Balanced algebras

Another family of examples comes from the observation that every \mathbb{R} -algebra A with an algebra involution τ is a $\mathbb{Z}/2$ -graded algebra, with $A_0 = A^\tau$ and $A_1 = \{a \in A : \tau(a) = -a\}$.

Definition 2.13. We will say that an algebra involution is *balanced* (or that A is balanced) if there is a unit u in A with $\bar{u} = -u$. This implies that there is a left A_0 -module isomorphism $A_0 \cong A_1$, $a \mapsto au$.

Remark. If A is simple, and there is a left A_0 -module isomorphism $A_0 \cong A_1$ sending 1 to u , then u must be a unit of A . Indeed, for each $b \in A_0$ there is a unique $a \in A_0$ such that $au = ub$. It follows that $uA_1 = uA_0u = A_1u$ and hence that Au is a 2-sided ideal of A . Since A is a simple algebra, we must have $Au = A$ and hence u is a unit of A .

The prototype of a balanced involution is the canonical involution (induced by -1 on V) on the Clifford algebra $A = C^{p,q}$ of a quadratic form on V of rank $p + q \geq 2$ and signature $q - p$. As in Example 2.10, A^0 may not be simple.

Example 2.14. A cannot be balanced if $\dim(A)$ is odd, and may not be balanced if $\dim(A)$ is even. For example, the algebra $A = M_4(\mathbb{R})$, with the involution defined by conjugation with the diagonal matrix $(1, 1, 1, -1)$, is not balanced: $A_0 = M_3(\mathbb{R}) \times \mathbb{R}$ and A_1 is the 6-dimensional subspace spanned by $\{e_{4j}, e_{j4} : j < 4\}$. By Theorem 1.8, $KR^A(X) \cong KO_G(X)$.

When A is balanced, the map $a \mapsto au$ defines an isomorphism $A \xrightarrow{\cong} A_\sigma$ of Real A -modules, where A_σ is defined in Example 2.8.

Theorem 2.15. *Let A be a Banach algebra with a balanced involution. If X is compact, any Real A -bundle E on X is a direct summand of a Real A -bundle of the form $X \times A^n$, with involution $\overline{(x, a)} = (\bar{x}, \bar{a})$.*

The Grothendieck group $KR^A(X)$ is isomorphic to $K_0(\Lambda)$, where Λ is the ring of continuous functions $f : X \rightarrow A$ satisfying $f(\bar{x}) = \overline{f(x)}$.

Finally, if the involution on X is trivial, $KR^A(X)$ is the usual Grothendieck group $K^{A_0}(X)$ of A_0 -bundles on X .

Proof. By Lemma 2.7, any Real A -bundle is a summand of a free Real A -bundle; by Lemma 2.4 and Example 2.8, free Real A -bundles have the form $X \times (A \times A_\sigma)^n$. Using the isomorphism $A \xrightarrow{\cong} A_\sigma$ of Real A -modules, it follows from Theorem 1.4 that every Real A -bundle is a summand of a trivial Real A -bundle $X \times A^m$.

The assertion about Λ comes from Theorem 1.7. If the involution is trivial on X , Λ is the ring of continuous functions $X \rightarrow A^0$. \square

Example 2.16 (Real quaternionic bundles). Theorem 2.15 applies to Real quaternion bundles (see Example 2.12). Indeed, conjugation by i is a balanced involution of $A = \mathbb{H}$ ($\bar{j} = -j$ and $\bar{k} = -k$).

If X has a trivial involution, then Theorem 2.15 shows that the category of Real quaternionic bundles is equivalent to the category of complex vector bundles on X . In particular, $KR^{\mathbb{H}}(X) \cong KU(X)$.

On the other hand, if X is $Y \times \{\pm 1\}$ with $\overline{(y, \varepsilon)} = (y, -\varepsilon)$, the group $KR^{\mathbb{H}}(X)$ is isomorphic to the symplectic K -theory $KSp(Y)$.

Remark. In [6], J. Dupont defined “symplectic bundles” which, in spite of the name, are not related to our constructions. His groups $Ksp_n(X)$ are just $KR_{n+4}(X)$. For instance, when X has a trivial involution, our group $KR^{\mathbb{H}}(X)$ is $KU(X)$ but Dupont’s group is $KR_4(X) = KO_4(X) = Ksp(X)$.

Example 2.17. Every Clifford algebra $C^{p,q}$ is canonically $\mathbb{Z}/2$ -graded, and the associated involution is balanced. Thus Theorem 2.15 applies when $C^{p,q}$ is a simple algebra. For example, consider the Clifford algebra $C^{0,2} \cong M_2(\mathbb{R})$; its fixed subalgebra is \mathbb{C} . As in Example 2.16, if X has a trivial involution, then $KR^A(X) \cong KU(X)$. However, if $X = Y \times \{\pm 1\}$ then $KR^A(X)$ is isomorphic to $KO(Y)$.

3. GROTHENDIECK-WITT GROUPS

Suppose that an \mathbb{R} -algebra A is a $*$ -algebra (i.e., $*$: $A \xrightarrow{\cong} A^{\text{op}}$ is an algebra isomorphism such that $a^{**} = a$ for all $a \in A$), and that τ is an algebra involution of A commuting with $*$. If \mathcal{A} is an algebra bundle on X with fibers A , and E is a Real \mathcal{A} -bundle, then the dual $E^* = \text{Hom}_A(E, A)$ is also a Real \mathcal{A} -bundle; if $f \in E_x^*$ then $af \in E_x^*$ is defined by $e \mapsto f(e)a^*$, and $(\tau f)(e) = \tau(f(e))$.

Definition 3.1. An ε -symmetric form ($\varepsilon = \pm 1$) on a Real \mathcal{A} -bundle E is an isomorphism $\psi : E \xrightarrow{\cong} E^*$ of Real \mathcal{A} -bundles such that the bilinear form $B(x, y) = \psi(x)y$ satisfies $B(y, x) = \varepsilon B(x, y)^*$. Note that $B(x, ay) = aB(x, y)$ but $B(ax, y) = B(x, y)a^*$.

The Grothendieck-Witt group ${}_{\varepsilon}GR^{\mathcal{A}}(X)$ is defined to be the Grothendieck group of the category of ε -symmetric forms (E, ψ) .

The groups $GR^{\mathcal{A}}(X) = {}_{+1}GR^{\mathcal{A}}(X)$, and to a lesser extent ${}_{-1}GR^{\mathcal{A}}(X)$, are the focus of our recent paper [13].

Remark. The definition of Grothendieck-Witt group in [17] includes the relation that $[(E, \varphi)] = [H(L)]$ whenever E has a *Lagrangian*, i.e., a subspace L with $L = L^{\perp}$. By Lemma 1.3 and [17, Lemma 2.9], this relation is redundant in our topological framework.

Here is a useful general principle.

Lemma 3.2. *Given a symmetric form ψ with bilinear form B , and an ε -symmetric form φ on E , $\theta = \psi^{-1}\varphi$ is ε -self-adjoint for B , i.e.:*

$$B(\theta x, y) = \varepsilon B(x, \theta y).$$

Proof. We have $B(\theta x, y) = \psi(\psi^{-1}\varphi x)y = \varphi(x)y$, and similarly $B(x, \theta y)$ equals $B(\theta y, x)^* = [\varphi(y)x]^* = \varepsilon \varphi(x)y$. \square

We can extend $*$ to $A \otimes \mathbb{C}$ by setting $(a \otimes z)^* = (a^*) \otimes \bar{z}$, where \bar{z} is the complex conjugate of z . There is a second anti-involution \dagger on $A \otimes \mathbb{C}$, defined by $(a \otimes z)^{\dagger} = (a^*) \otimes z$. If E is a finitely generated projective $A \otimes \mathbb{C}$ -module, we write E^* and E^{\dagger} for its dual $\text{Hom}_{A \otimes \mathbb{C}}(E, A \otimes \mathbb{C})$, endowed with the respective anti-involutions $*$ and \dagger . These left module structures define $(a \otimes z)f$ to be $e \mapsto f(e)(a^* \otimes \bar{z})$ for $f \in E^*$, and $e \mapsto f(e)(a^* \otimes z)$ for $f \in E^{\dagger}$.

A *Hermitian form* on E is a map $\psi : E \rightarrow E^*$ which is symmetric for the anti-involution $*$. That is, its associated form $\langle x, y \rangle = \psi(x)(y)$ satisfies $\langle y, x \rangle = \langle x, y \rangle^*$ and $\langle (a \otimes z)x, y \rangle = \langle x, y \rangle(a^* \otimes \bar{z})$.

Lemma 3.3. *If φ is an ε -symmetric form on E for \dagger and ψ is Hermitian then $\theta = \psi^{-1}\varphi$ is \mathbb{C} -antilinear, and $\langle \theta x, y \rangle = \varepsilon \overline{\langle x, \theta y \rangle}$, where $\overline{a \otimes z}$ denotes $a \otimes \bar{z}$.*

Proof. The proof is similar to that of Lemma 3.2: $\langle \theta x, y \rangle = \varphi(x)y$ as before, and

$$\langle x, \theta y \rangle = \langle \theta y, x \rangle^* = [\varphi(y)x]^* = \varepsilon[\varphi(x)y]^{\dagger*} = \varepsilon \overline{\langle \theta x, y \rangle}.$$

The fact that θ is \mathbb{C} -antilinear follows from the \mathbb{C} -linearity of φ and the \mathbb{C} -antilinearity of ψ . \square

We first consider the following class of examples. Recall from [18] that a C^* -algebra over \mathbb{R} is a Banach $*$ -algebra over \mathbb{R} , $*$ -isometrically isomorphic to a norm-closed subalgebra of linear operators on a real Hilbert space. We extend the C^* -structure of A (as an \mathbb{R} -algebra) to a C^* -structure of $A \otimes \mathbb{C}$ (as a \mathbb{C} -algebra) by setting $(a \otimes z)^* = (a^*) \otimes \bar{z}$.

Example 3.4. Suppose that A^0 is a C^* -algebra over \mathbb{R} . The complex C^* -algebra $A = A^0 \otimes \mathbb{C}$ is equipped with the involution τ coming from complex conjugation. The structure map $A \rightarrow A^{\text{op}}$ is $(a_0 \otimes z) \mapsto a_0^* \otimes z$ (the \dagger map defined before Lemma 3.3); as it commutes with τ , we can consider the groups ${}_{\varepsilon}GR^A(X)$.

In this case we consider the two auxilliary \mathbb{R} -algebras

$$A^+ = A^0 \otimes_{\mathbb{R}} M_2(\mathbb{R}) \quad \text{and} \quad A^- = A^0 \otimes_{\mathbb{R}} \mathbb{H}.$$

That is, A^+ is generated by $A = A^0 \otimes \mathbb{C}$ and an element j satisfying $j^2 = 1$ and $ij = -ji$, while A^- is generated by A and an element j satisfying $j^2 = -1$ and $ij = -ji$. In both cases, j commutes with A . The involution on A induces an involution on A^{\pm} fixing A^0 and j .

If \mathcal{A}^0 is an algebra bundle with fiber A^0 , we can define algebra bundles \mathcal{A} and \mathcal{A}^{\pm} in an obvious way.

Theorem 3.5. *If $\mathcal{A} = \mathcal{A}^0 \otimes \mathbb{C}$, with involution $\tau(a \otimes z) = a \otimes \bar{z}$, and anti-involution $(a \otimes z) \mapsto a^* \otimes z$, we have canonical isomorphisms*

$${}_{+1}GR^{\mathcal{A}}(X) \cong KR^{\mathcal{A}^+}(X) \quad \text{and} \quad {}_{-1}GR^{\mathcal{A}}(X) \cong KR^{\mathcal{A}^-}(X).$$

Proof. We give the proof for $\varepsilon = +1$; the proof for $\varepsilon = -1$ is similar. Let E be a Real A -bundle provided with a symmetric bilinear form φ . Choose a Hermitian metric ψ on E compatible with the involution. (Hermitian metrics exist by [11, 2.7]; $(\psi(x) + \psi(\tau x))/2$ is compatible with τ .) Then φ is associated to a self-adjoint invertible operator θ on E by Lemma 3.2 and Lemma 3.3; by construction, θ commutes with the involution and is complex conjugate linear.

As it is self-adjoint, all eigenvalues of θ are real and positive. Changing the metric ψ up to homotopy, we may even assume that θ is unitary. Since $\theta = \theta^*$ we have $\theta^2 = 1$. Setting $j = i\theta$, we have $j^2 = 1$ and $ij = -ji$ (as $i\theta = -\theta i$). Thus E is a Real \mathcal{A}^+ bundle.

Conversely, given a Real \mathcal{A}^+ bundle E on X , choose a G -invariant Hermitian metric ψ on E whose bilinear form satisfies $\langle jx, y \rangle = \langle x, jy \rangle$. Setting $\theta = ji$, $\varphi = \psi\theta$ is a symmetric bilinear form. By inspection, the map between K -groups so obtained is inverse to the map defined in the first paragraph. \square

Corollary 3.6. *The theory $KR^{A^+}(X)$ is canonically isomorphic to the usual equivariant K -theory $K_G^{A^0}(X)$, where G is acting trivially on \mathcal{A}^0 . Therefore, we have an isomorphism*

$$GR^{A^+}(X) \cong K_G^{A^0}(X).$$

Proof. For $\varepsilon = +1$, the involution on $A^+ = A^0 \otimes_{\mathbb{R}} M_2(\mathbb{R})$ is conjugation by the diagonal matrix $j = (1, -1)$. Exactly as in Example 2.10, Theorem 1.8 yields $KR^{A^+}(X) \cong K_G^{A^0}(X)$. \square

Despite the symmetry of Theorem 3.5, the theory for $\varepsilon = -1$ is quite different for $\varepsilon = +1$. Indeed, the algebra \mathcal{A}^- is $\mathcal{A}^0 \otimes_{\mathbb{R}} \mathbb{H}$, where G acts trivially on \mathcal{A}^0 and by the quaternionic involution $(i \mapsto -i, j \mapsto -j)$ on \mathbb{H} . That is, τ is conjugation by $k = ij$. Note that Theorem 1.8 does not apply because $k^2 \neq 1$. Indeed, the group $KR^{A^-}(X)$ is different from $KSp_G(X)$ in general.

Example 3.7. The group $GR(X)$ discussed in [13] is the special case $GR^{\mathbb{C}}(X)$, where G acts on $A = \mathbb{C}$ by conjugation and $A^0 = \mathbb{R}$. by Corollary 3.6 we have $GR(X) \cong KO_G(X)$.

On the other hand, the group ${}_{-1}GR(X)$ is isomorphic to $KR^{\mathbb{H}}(X)$ by Theorem 3.5, where τ is conjugation by i (as in Example 2.16). This seems to be a new example of a twisted K -group.

Now suppose that A is a C^* -algebra over \mathbb{R} , with a G -action compatible with the C^* -structure, so that we can define ${}_{\varepsilon}GR^A(X)$.

Theorem 3.8. *If A is a C^* -algebra over \mathbb{R} , and G acts compatibly with $*$, we have canonical isomorphisms*

$${}_{\varepsilon}GR^A(X) \cong KR^{A'}(X),$$

where A' is the algebra bundle $\mathcal{A} \otimes_{\mathbb{R}} \mathbb{R}[t]/(t^2 = \varepsilon)$.

Note that $A' \cong \mathcal{A} \times \mathcal{A}$ if $\varepsilon = +1$, while $A' \cong \mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}$ if $\varepsilon = -1$.

Proof. We indicate the modifications to the proof of Theorem 3.5. Given (E, φ) , we choose a Riemannian metric ψ (instead of a Hermitian metric), compatible with the involution, and form the ε -self-adjoint operator θ using Lemma 3.2. Changing the metric up to homotopy, we may assume that θ is orthogonal, so $\theta^2 = \varepsilon I$. Thus E is a Real $\mathcal{A}[\theta]$ -bundle, where $a\theta = \theta a$.

Conversely, given a Real $\mathcal{A}[\theta]$ -bundle E , choose a G -invariant Riemannian metric ψ on E whose bilinear form satisfies $\langle \theta x, y \rangle = \langle x, \theta y \rangle$. Then $\varphi = \psi\theta$ is a symmetric form on E . This proves that $GR^A(X) \cong KR^{A[\theta]}(X)$. \square

Examples 3.9. (a) if $A = M_n(\mathbb{R})$ and $*$ is matrix transpose, then the involution can be conjugation by an orthogonal matrix x . If $x^2 = 1$ then $GR^A(X) \cong KR_G^A(X) \cong KO_G(X)$ by Example 1.9.

(b) if $A = M_n(\mathbb{C})$ and $*$ is conjugate transpose, then the involution can be conjugation by a unitary matrix x . Here $GR^A(X) \cong KU_G(X)$.

(c) if $A = M_n(\mathbb{H})$ and $*$ is matrix transpose composed with $\mathbb{H} \cong \mathbb{H}^{\text{op}}$, then the involution can be conjugation by a symplectic matrix x . In this case, $GR^A(X) \cong KSp_G(X)$, again by Example 1.9.

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